指定された値のまわりの級数解を求めよ.

(1) $(1-x)y'-y=0,\quad y(0)=2 \qquad (x=0)$

(2) $y''+x^2y=x,\ y(0)=a, y'(0)=b \qquad (x=0,$ x^9の項まで求めるだけでよい.)

(1) $(1-x)y'-y=0,\ y(0)=2$

$y=\sum_{m=0}^{\infty}a_m x^m=a_0+a_1x+a_2x^2+\cdots$ と置く

$$y'=\sum_{m=1}^{\infty}ma_m x^{m-1}=\sum_{m=0}^{\infty}(m+1)a_{m+1}x^m$$

与式に代入

$$\left\{\sum_{m=0}^{\infty}(m+1)a_{m+1}x^m-\sum_{m=1}^{\infty}ma_m x^m\right\}-\sum_{m=0}^{\infty}a_m x^m=0$$

$$\Leftrightarrow\sum_{m=1}^{\infty}\left\{(m+1)a_{m+1}x^m-(m+1)a_m x^m\right\}+a_1-a_0=0$$

$$\Leftrightarrow\sum_{m=0}^{\infty}(m+1)(a_{m+1}-a_m)x^m=0$$

よって、$a_{m+1}-a_m=0\Leftrightarrow a_{m+1}=a_m$ （mは0以上の整数）

$a_0=a_1=a_2=\cdots=a_m$ なので、

$$y=a_0\sum_{m=0}^{\infty}x^m$$

$y(0)=2$ より $a_0=2$

$$\therefore y=2\sum_{m=0}^{\infty}x^m=\frac{2}{1-x}$$

(2) $y'' + x^2 y = x \qquad y(0) = a, y'(0) = b$

$y = \sum_{m=0}^{\infty} c_m x^m$ と置くと，

$$y = \sum_{m=2}^{\infty} c_{m-2} x^{m-2}, \quad y'' = \sum_{m=2}^{\infty} m(m-1) c_m x^{m-2} = \sum_{m=0}^{\infty} (m+2)(m+1) c_{m+2} x^m$$

与式に代入

$$\sum_{m=0}^{\infty} (m+2)(m+1) c_{m+2} x^m + x^2 \sum_{m=2}^{\infty} c_{m-2} x^{m-2} = x$$

$$\Leftrightarrow 2 \times 1 \times c_2 x^0 + 3 \times 2 \times c_3 x^1 + \sum_{m=2}^{\infty} (m+2)(m+1) c_{m+2} x^m + \sum_{m=2}^{\infty} c_{m-2} x^m - x = 0$$

$$\Leftrightarrow 2c_2 + (6c_3 - 1)x + \sum_{m=2}^{\infty} \left\{ (m+2)(m+1) c_{m+2} + c_{m-2} \right\} x^m = 0$$

よって，

$$c_2 = 0, \quad c_3 = \frac{1}{6}, \quad c_{m-2} = -(m+2)(m+1) c_{m+2} \Leftrightarrow c_m = -(m+4)(m+3) c_{m+4}$$

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(2) の続き

$y(0)=a,\quad y'(0)=b$ 　より、

$y(0)=c_0=a,\quad y'(0)=c_1=b$

$$c_4=-\frac{c_0}{4\times 3}=-\frac{a}{12},\quad c_5=-\frac{c_1}{5\times 4}=-\frac{b}{20},$$

$$c_6=-\frac{c_2}{6\times 5}=0,\quad c_7=-\frac{c_3}{7\times 6}=-\frac{1}{252},$$

$$c_8=-\frac{c_4}{8\times 7}=\frac{a}{672},\quad c_9=-\frac{c_5}{9\times 8}=\frac{b}{1440}$$

$$\therefore y=a+bx+\frac{1}{6}x^3-\frac{a}{12}x^4-\frac{b}{20}x^5-\frac{1}{252}x^7+\frac{a}{672}x^8+\frac{b}{1440}x^9+\dots$$