Measurement errors and uncertainties

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Outline

- Error analysis main goals
- Error types and origins
  - Random vs. systematic errors
  - Instrumental uncertainties, statistical uncertainties, miscalibration errors
- Uncertainty due to random errors
  - Gaussian (normal) distribution
  - Student's *t* distribution
  - Binomial distribution
  - Poisson distribution
- Uncertainty due to systematic errors
- Total uncertainty
- Uncertainty of a result
- Covariance and correlation

Basic concepts and terms

- Variables
  - Experimental tests are performed to answer a question. Once the question is defined, we need to identify the relevant process parameters and variables. Variables are quantities that influence the test.
  - An independent variable can be changed independently of other variables
  - A dependent variable is affected by changes in one or more other variables.
- Controlled variables
  - A variables is controlled if it can be held at a constant value or at some prescribed condition during a measurement.

Basic concepts and terms

- Uncontrolled variables
  - Variables that are not or cannot be controlled during measurement, but affect the value of the variable measured.
  - Their influence can confuse the clear relation between cause and effect in a measurement.

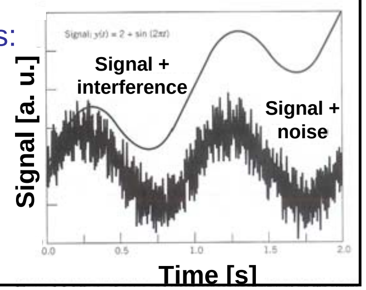
Effect of uncontrolled variables:

-Interference

impose a false trend

-Noise

increase data scatter



Basic concepts and terms

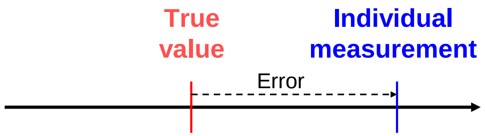
- Measurable Quantity
  - A property of phenomena, bodies, or substances that can be expressed quantitatively (e.g. length, mass, time…)
- Measurement
  - The process of finding the value of a quantity.
- True value of a measurable quantity
  - The actual value of the quantity being measured
- Measurement error
  - The deviation of the measurement from the true value
- Uncertainty
  - Interval within which the true value of the measured quantity lies with a given probability

Errors

- Errors are not mistakes
  - It is impossible to completely eliminate them.
- Repeat the same measurements several times
  - The spread in your measured values gives a valuable indication of the uncertainty in your measurements (take a sample). Only valid for random errors.
- Type of errors: random / systematic
- Source of errors: instrumental errors, statistical errors, miscalibration

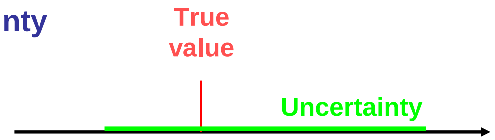
Error and uncertainty

Error



True value: Actual value of the parameter being measured
Error: Single value which cannot be known exactly; idealized concept

Uncertainty



Uncertainty: Interval that has a pre-assigned probability of containing the true value; cannot be used to correct a measurement result

Error analysis main goals

All measurements are subject to uncertainties

=> Error analysis is the study and evaluation of these uncertainties

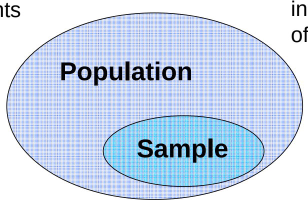
-How large are uncertainties?

-How to reduce uncertainties?

Population and sample

Population: the set of all possible measurements of a parameter

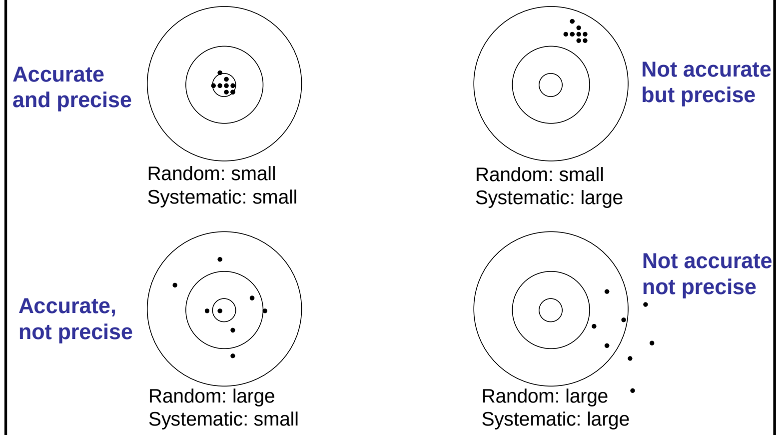
Sample: the set of individual measurements of a parameter



Samples must be chosen so that they are representative of the population

When the sample is not representative of the entire population, the sample is said to be biased and can lead to wrong statements about the population.

Random and systematic errors



The sources of uncertainties: Instrumental uncertainties

Fluctuations in readings due to imperfection in the equipment (lack of precision), surrounding noise, etc.

Examples:
-Number of bits used in an ADC
-Fluctuations in the power supply
-Effect of cables
-Effect of thermal noise (Johnson noise)

The sources of uncertainties: Statistical uncertainties

Number of counts in a detector per unit time for a random process (e.g. photons hitting a detector, shot noise in electronic device)

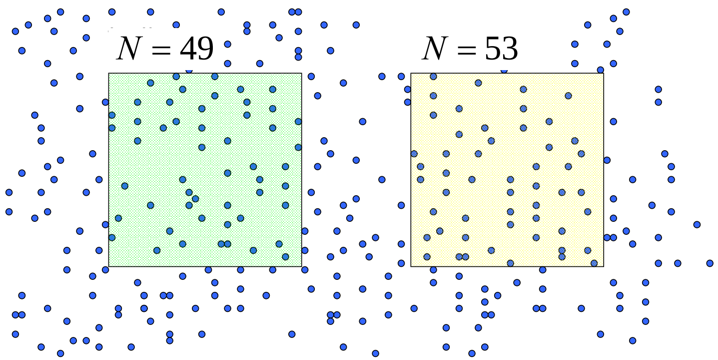
Origin

- Statistical uncertainties arise from statistical fluctuations (random process) in the collections of numbers of counts over finite intervals of time.
- Not related to a lack of precision in the measuring instruments!

Properties

- The standard deviation is the root of the mean: $\sigma = \sqrt{\mu}$
- The observed values are distributed according to a Poisson distribution.

Statistical fluctuations in the number of counts



The sources of uncertainties: Systematic errors

- Difficult to detect; check the measuring device against a device known to be more reliable.
- Systematic uncertainties cannot be treated statistically; random uncertainties can be treated statistically.
- Most common causes of systematic errors:
  imperfect calibration corrections,
  imperfect data acquisition systems,
  imperfect data reduction techniques

Uncertainty due to random errors

- The Gaussian distribution
- The Student's *t* distribution
- The Poisson distribution
- The Chi-square distribution

Statistical treatment of random uncertainties distributed according to the Gaussian distribution

Statistical treatment of random uncertainties: Main results

Hypothesis: systematic errors are negligible and random errors are small
N measurements $x_1,\ldots, x_N$ of the same quantity X

The best estimate of the quantity X — $\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$

The sample standard deviation — $\sigma_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \bar{x}\right)^2}$

The standard deviation of the mean — $\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$

The value of X — $\bar{x} \pm 2\sigma_{\bar{x}}$
(Expect 95% of any measurements of X to fall in the range $\bar{x} \pm 2\sigma_{\bar{x}}$)

Generally, $\bar{x} \pm t\sigma_{\bar{x}}$ with $P(\text{within } t\sigma_{\bar{x}})$

Histograms: Bar histogram
(Handle and display discrete measurements)

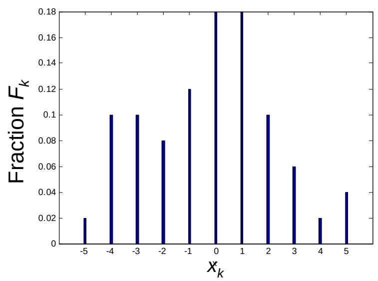
Integer values x_k, with n_k the number of times it occured

$$\sum_k n_k = N$$

fraction $F_k = \frac{n_k}{N}$

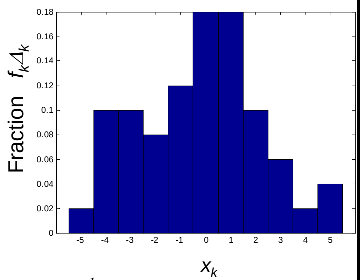
$$\sum_k F_k = 1 \text{ (normalization condition)}$$

$$\bar{x} = \sum_k x_k F_k$$



In the absence of systematic error, we may assume that the histogram peak is somewhere near the true value X. The spread about the peak gives an indication of the precision of the measurements.

Histograms: Bin histogram
(Handle and display measurements)



N measurements $x_1,\ldots,x_N$ of x

Δ_k bin size

count how many values fall in each bin

$f_k\Delta_k = n_k/N$ fraction of measurements in the k^{th} bin

$$\sum_k f_k\Delta_k = 1 \text{ (normalization condition)}$$

$$\bar{x} = \sum_k x_k f_k \Delta_k$$

Limiting distributions

Limiting distribution: The distribution of results that would be obtained if the number of measurements becomes infinitely large

N measurements $x_1,\ldots,x_N$ of x, with $N \to \infty$

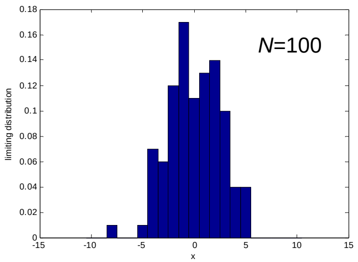
$f(x)\,dx$ fraction of measurements that falls in $[x, x+dx]$

$f(x)\,dx$ probability that any one measurement falls in $[x, x+dx]$

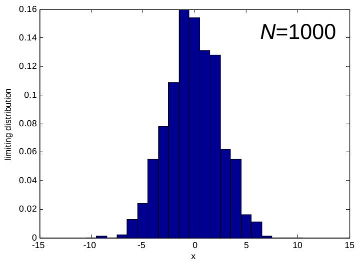
$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \text{ (normalization condition)}$$

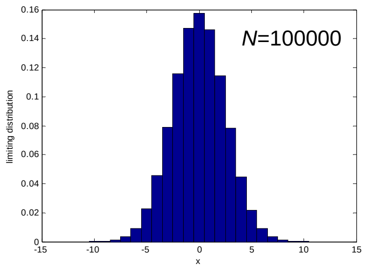
Mean value expected after many measurements: $\bar{x} = \int_{-\infty}^{\infty} x\, f(x)\, dx = X$

Mathematical expectation of the random variable x is $E(x) = \bar{x} = \int_{-\infty}^{\infty} x\, f(x)\, dx$



The limiting distribution is a theoretical construct that can never be measured exactly.





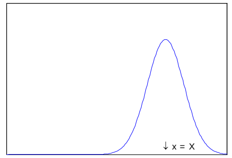
The normal distribution

Statistical analysis of repeated measurements

Measurements of *X* are subject to many small random errors and negligible systematic errors

- Measurements are distributed in accordance with a bell-shaped curve, centered on the true value of *X* (we assume that a true value exists).
- The mathematical function that describes the bell-shaped curve is called the normal distribution or Gauss function:

$$e^{-\frac{1}{2}\left(\frac{x-X}{\sigma}\right)^2}$$



Normal distribution: Normalization

Normalization condition : $\int_{-\infty}^{\infty} f(x)dx = 1$

$$f(x) = Ce^{-\frac{1}{2}\left(\frac{x-X}{\sigma}\right)^2}$$

$$1 = \int_{-\infty}^{\infty} f(x)dx = C\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-X}{\sigma}\right)^2} dx$$

$u = \frac{1}{\sqrt{2}}\left(\frac{x-X}{\sigma}\right)$ and get

$$1 = C\sigma\sqrt{2}\int_{-\infty}^{\infty} e^{-u^2} du = C\sigma\sqrt{2\pi} \quad \left(\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}\right)$$

$$C = \frac{1}{\sigma\sqrt{2\pi}}$$

Normal distribution

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-X}{\sigma}\right)^2}$$

Proof of $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$

$$I = \int_{-\infty}^{\infty} e^{-u^2} du$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \times \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy$$

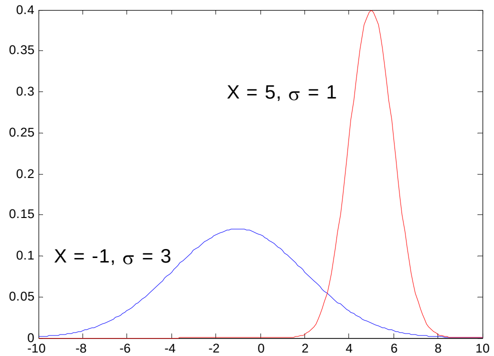
coordinate transformation $x = r\cos\theta, y = r\sin\theta$

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} rdrd\theta = 2\pi\int_0^{\infty} e^{-r^2} rdr = 2\pi\left(\frac{e^{-r^2}}{-2}\right)\Bigg|_0^{\infty}$$

$$I^2 = \pi$$

$$I = \sqrt{\pi}$$

Normal distribution: True value *X* and width σ



Normal distribution: expected average

Expected average for the distribution $f(x)$: $\bar{x} = \int_{-\infty}^{\infty} x f(x) dx$

(Note that for symmetry reasons, the average should be X)

$$\bar{x} = \int_{-\infty}^{\infty} x f(x) dx = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-X}{\sigma}\right)^2} dx$$

change of variables $u = \frac{1}{\sqrt{2}}\left(\frac{x-X}{\sigma}\right)$

$$\bar{x} = \int_{-\infty}^{\infty} (\sigma\sqrt{2}u + X) \frac{1}{\sigma\sqrt{2\pi}} e^{-u^2} \sigma\sqrt{2} du = \frac{1}{\sqrt{\pi}}\left(\int_{-\infty}^{\infty} \sigma\sqrt{2} u e^{-u^2} du + X \int_{-\infty}^{\infty} e^{-u^2} du\right)$$

$$\left(\int_{-\infty}^{\infty} u e^{-u^2} du = 0 \text{ odd function; } \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}\right)$$

$$\bar{x} = X$$

Normal distribution: Expected standard deviation

Expected standard deviation for the distribution $f(x)$:

$$\sigma_x{}^2 = \int_{-\infty}^{\infty} (x-X)^2 f(x) dx$$

$$\sigma_x{}^2 = \int_{-\infty}^{\infty} (x-X)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-X}{\sigma}\right)^2} dx$$

change of variables $u = \frac{1}{\sqrt{2}}\left(\frac{x-X}{\sigma}\right)$

$$\sigma_x{}^2 = \int_{-\infty}^{\infty} 2\sigma^2 u^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-u^2} \sigma\sqrt{2} du = -\frac{\sigma^2}{\sqrt{\pi}}\left(\int_{-\infty}^{\infty} (-2ue^{-u^2}) u du\right)$$

$$\sigma_x{}^2 = -\frac{\sigma^2}{\sqrt{\pi}}\left((ue^{-u^2})\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-u^2} 1 du\right) \text{ (integration by parts)}$$

$$\left(ue^{-u^2}\Big|_{-\infty}^{\infty} = 0 \text{ odd function; } \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}\right)$$

$$\sigma_x{}^2 = \sigma^2$$

Normal distribution: Probability that *X* - σ ≤ *x* ≤ *X* + σ

Probability that a measurement will fall in $X\text{-}\sigma \le x \le X + \sigma$:

$$P(\text{within } \sigma) = \int_{X-\sigma}^{X+\sigma} f(x) dx$$

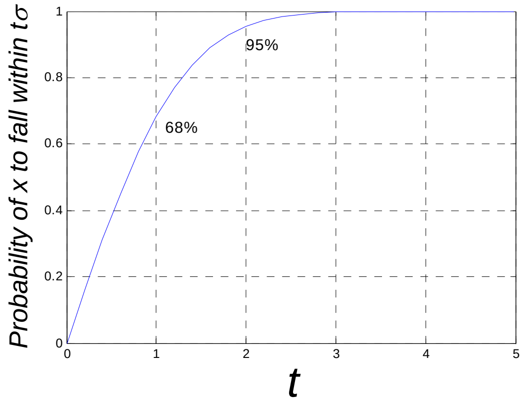
$$P(\text{within } \sigma) = \int_{X-\sigma}^{X+\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-X}{\sigma}\right)^2} dx$$

change of variables $u = \left(\frac{x-X}{\sigma}\right)$

$$P(\text{within } \sigma) = \int_{-1}^{1} \frac{1}{\sigma\sqrt{2\pi}} e^{-u^2/2} \sigma du = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-u^2/2} du \approx 0.68$$

$$P(\text{within } t\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-u^2/2} du$$

Probability that a measurement falls within *t* standard deviation of the true value



This function is often referred to as the error function $erf\left(t/\sqrt{2}\right)$

How to report uncertainties

N measurements $x_1,...,x_N$ of a quantity X :

value of x $= \bar{x} \pm \sigma_{\bar{x}}$ (68% of measurements fall in $\bar{x} \pm \sigma_{\bar{x}}$)

value of x $= \bar{x} \pm 2\sigma_{\bar{x}}$ (95% of measurements fall in $\bar{x} \pm 2\sigma_{\bar{x}}$)

When reporting uncertainties, two quantities should be given :

- a range (interval) and
- the corresponding confidence level

Example : $x = 23.07 \pm 0.02$ at the 95% confidence level.

Note : $\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$, $\sigma_{\bar{x}} \neq \sigma_x$

Acceptability of a measurement

Measurements of a quantity x : $x_{\text{best}} \pm \sigma$

where x_{best} is our best estimate of x,

σ is the standard deviation.

Expected value is x_{exp} (known from theory or reference measurements)

x_{exp} differs from x_{best} by t standard deviations : $t = \frac{|x_{\text{best}} - x_{\text{exp}}|}{\sigma}$

The probability that a measurement fall outside $t\sigma$ is $P(\text{outside } t\sigma)$

If this probability is unreasonably small, the measurement is unacceptable.

If $P(\text{outside } t\sigma$) is less than some chosen level,

the agreement is unacceptable at that level.

Example : If $P(\text{outside } t\sigma) < 5\%$, the agreement is unacceptable at the 5% level.

Best estimates for *X* and σ

(1/2)

Best estimates of X and σ based on the N measured values.

The N values are normally distributed.

$$P(x_1) = P(x \in [x_1, x_1 + dx_1]) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_1 - X}{\sigma}\right)^2} dx_1$$

$$P(x_1) \propto \frac{1}{\sigma} e^{-\frac{1}{2}\left(\frac{x_1 - X}{\sigma}\right)^2}$$

$$P(x_N) \propto \frac{1}{\sigma} e^{-\frac{1}{2}\left(\frac{x_N - X}{\sigma}\right)^2}$$

The probability that we observed the whole set $x_1,...,x_N$ is

$$P_{X,\sigma}(x_1,...,x_N) = P(x_1) \times ... \times P(x_N) \propto \frac{1}{\sigma^N} e^{-\frac{1}{2}\sum_i \left(\frac{x_i - X}{\sigma}\right)^2}$$

The best estimates for X and σ are those values for which $P_{X,\sigma}(x_1,...,x_N)$ is maximum (the principle of maximum likelihood)

Best estimates for *X* and σ

(2/2)

$$P_{X,\sigma}(x_1,...,x_N) \propto \frac{1}{\sigma^N} e^{-\frac{1}{2}\sum_i\left(\frac{x_i-X}{\sigma}\right)^2}$$

The best estimates for X and σ are those values for which $P_{X,\sigma}(x_1,...,x_N)$ is maximum

Best estimate of X : X for which $\sum_i\left(\frac{x_i-X}{\sigma}\right)^2$ is minimum :

minimum corresponds to $\frac{\partial\sum_i\left(\frac{x_i-X}{\sigma}\right)^2}{\partial X}=0$

$\sum_i(x_i-X)=0 \quad \Leftrightarrow \quad \mathrm{X}=\frac{1}{N}\sum_i x_i$

Best estimate of σ : $\sqrt{\frac{1}{N\text{-}1}\sum_{i=1}^{N}\left(x_i-\bar{x}\right)^2}$ (no proof)

Standard deviation of the mean

N measurements $x_1,...,x_N$ of a quantity x :

- the best estimate of the true value X is $\bar{x}=\frac{x_1+...+x_N}{N}$
- the uncertainty in this estimate is $\sigma_{\bar{x}}=\sigma_x/\sqrt{N}$

Many determinations of the average $\bar{x}$

$\bar{x}$ is normally distributed as the x_i :

- the true value for the $\bar{x}$ distribution is X
- the width of the $\bar{x}$ distribution is $\sigma_{\bar{x}}$

$$\sigma_{\bar{x}}=\sqrt{\left(\frac{\partial\bar{x}}{\partial x_1}\sigma_{x_1}\right)^2+...+\left(\frac{\partial\bar{x}}{\partial x_N}\sigma_{x_N}\right)^2}=\sqrt{N\left(\frac{1}{N}\sigma_x\right)^2}=\sigma_x/\sqrt{N}$$

(Measurements of the same quantity $x \Rightarrow \sigma_{x_1}=...=\sigma_{x_N}=\sigma_x$)

Propagation of uncertainties: Main results

(1/2)

The N measurements of the quantity x are normally distributed about the true value X, with width σ.

We calculate the quantity $q=x+C^{st}$:
The calculated values of q are normally distributed about the true value $X+C^{st}$, with width σ.

We calculate the quantity $q=\alpha\times x$:
The calculated values of q are normally distributed about the true value $\alpha\times X$, with width $\alpha\times\sigma$.

Propagation of uncertainties: Main results

(2/2)

Consider measurements of the independent quantities $x, y,..., z$, normally distributed about their true values $X, Y,..., Z$, with width $\sigma_x, \sigma_y,...., \sigma_z$.

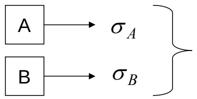
We calculate the quantity $q=x+y+...+z$:
The calculated values of q are normally distributed about the true value $X+Y+...+Z$, with width $\sqrt{\sigma_x^2+\sigma_y^2+....+\sigma_z^2}$.

(1/3)

Combining measurements from two sensors of different accuracies

Sensors A and B, measuring a quantity Z,
produce measurements Z_A and Z_B whose
errors have zero mean and variances σ_A^2 and σ_B^2

Can the measurements from A and B be combined to produce a measurement that, on average, is statistically superior to either of the two original ones?



$f(A,B)$ with $\sigma_f < \sigma_A$ and $\sigma_f < \sigma_B$

(2/3)

Combining measurements …

We form a weighted measurement $Z_C = \alpha\, Z_A + \beta\, Z_B$

Z_C estimate of $Z \Rightarrow \beta = 1-\alpha$

$Z_C = \alpha\, Z_A + (1-\alpha) Z_B$

$\Leftrightarrow \sigma_C^2 = \alpha^2\, \sigma_A^2 + (1-\alpha)^2\, \sigma_B^2$

Minimize $f(\alpha) = \alpha^2\, \sigma_A^2 + (1-\alpha)^2\, \sigma_B^2$

$$\begin{cases} f'(\alpha) = 0 \Leftrightarrow \alpha = \sigma_B^2 / (\sigma_A^2 + \sigma_B^2) \\ f''(\alpha) > 0 \end{cases}$$

$$\sigma_C^2 = \frac{1}{\sigma_A^{-2} + \sigma_B^{-2}}$$

Z_C has on average a higher accuracy (smaller variance)

(3/3)

Combining measurements: Generalization to weighted averages

N separate measurements of the same quantity X

$x_1 \pm \sigma_1,\ x_2 \pm \sigma_2, \ldots,\ x_N \pm \sigma_N$

The best estimate is :

$$x_{best} = \frac{\sum_{i=1}^{N} x_i / \sigma_i^2}{\sum_{i=1}^{N} 1/\sigma_i^2}$$

Best estimate uncertainty :

$$\sigma_{best}^2 = \frac{1}{\sum_{i=1}^{N} 1/\sigma_i^2}$$

Statistical treatment of random uncertainties distributed according to the Student *t* distribution

Student's *t* distribution

Used in characterizing the mean and the standard deviation of the mean when the sample size is small (*N* < 20); the mean and our estimate of the standard deviation are poorly determined.

$$t = \frac{\bar{x} - X}{\sigma_{\bar{x}}}; \ \nu = N - 1 \text{ (degree of freedom)}$$

$$\text{Probability density function}(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\ \Gamma(\nu/2)(1 + x^2/\nu)^{(\nu+1)/2}}$$

where Γ is the Gamma function

The Student *t* distribution depends on *N* ($\nu = N - 1$) and converges to the Gaussian distribution as *N* goes to infinity.

Student's *t*-test

Test the agreement between observed and expected mean

N measurements having mean $\bar{x}$ and std deviation σ_x

Test the hypothesis that the true value is X.

Form the test function $t = \dfrac{|\bar{x} - X|}{\sigma_x / \sqrt{N}}$

	Confidence interval	
N-1	95%	99%
10	2.28	3.17
120	1.98	2.62
∞	1.96	2.58

t follows a Student's t distribution

where the samples of X are normally distributed

with parameter the number of degrees of freedom $\cong N - 1$

$X = \bar{x} \pm t\sigma_x / \sqrt{N}$ at the confidence level $P_{Student}$ (within $t\sigma_x / \sqrt{N}$)

The probability that the true value X lies within the limit is $P_{Student}$ (within $t\sigma_x / \sqrt{N}$)

Statistical treatment of random uncertainties distributed according to the Binomial distribution

Apply when the result is one of a small number of possible final states such as "heads or tails" process. A Process which gives discrete values.

The Bernoulli process

- An experiment often consists of repeated trials, each with two possible outcomes (success or failure)
  - Application example: testing of items as they come off an assembly line, where each test indicates a defective or a nondefective item.
- A Bernoulli process possesses the following properties
  - The experiment consists of *n* repeated trials
  - Each trial results in an outcome that may be classified as a success or a failure
  - The probability of success *p* remains constant from trial to trial
  - The repeated trials are independent
- The random variable described by a Bernoulli process follows a binomial distribution

Binomial distribution

$$P_{Binomial}(\nu \text{ successes in } n \text{ trials}) = B_{n,p}(\nu)$$
$$= \frac{n(n-1)\ldots(n-\nu+1)}{1\times 2\times\ldots\times\nu} p^{\nu} q^{n-\nu}$$

where

p is the probability of success in one trial

$q = 1 - p$ is the probability of failure in one trial

Properties

$$\overline{\nu} = \sum_{\nu=0}^{\infty} \nu\, B_{n,p}(\nu) = np$$

$$\sigma_{\nu} = \sqrt{np(1-p)} = \sqrt{npq}$$

Bernoulli process example

Three items are selected at random from a manufacturing process, inspected, and classified as defective or nondefective.

A defective item is designed a success. The number of successes is a random variable X.

The items are selected independently from a process that produces 25% defectives $(p = 0.25)$.

Outcome	x
NNN	0
NDN	1
NND	1
DNN	1
NDD	2
DND	2
DDN	2
DDD	3

$$P(NDN) = P(N)P(D)P(N) = \frac{3}{4}\frac{1}{4}\frac{3}{4} = \frac{9}{64}$$

x	0	1	2	3
$f(x)$	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

The binomial distribution of the discrete random variable *x*

$$B_{n=3,p=0.25}(x=1) = 3\times 0.25^{1}\times(1-0.25)^{3-1} = \frac{27}{64}$$

Statistical treatment of random uncertainties distributed according to the Poisson distribution

The sources of uncertainties: Statistical uncertainties

Number of counts in a detector per unit time for a random process

=> Statistical uncertainties arise from statistical fluctuations (random process) in the **collections of numbers of counts over finite intervals of time**.

Not related to a lack of precision in the measuring instruments.

Poisson distribution

$$P_{Poisson}(\nu \text{ counts in a given interval } t) = P_{\mu t}(\nu)$$

$$= e^{-\mu t}\frac{(\mu t)^{\nu}}{\nu!}$$

Properties

μ is the average number of outcomes per unit time

$$\sigma_{\mu t} = \sqrt{\mu t}$$

Statistical treatment

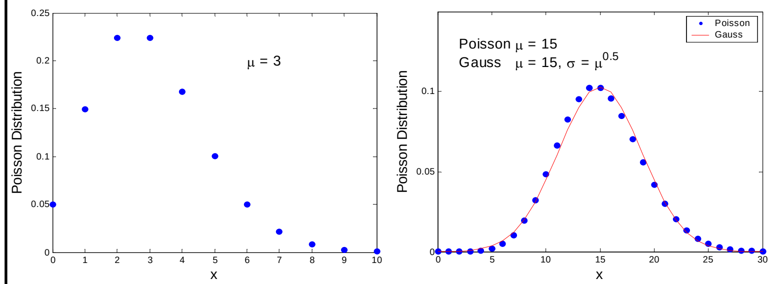
Usually cannot repeat measurements:

Estimate of the standard deviation of a single measurement is taken as $\sigma = \sqrt{\nu}$

If we make one measurement of the number of events in a defined time interval and get the answer ν

$$\nu \pm \sqrt{\nu}$$

Gaussian approximation to the Poisson distribution



Notes:

-The Poisson distribution is only defined at integer values (number of counts).

-The Gauss distribution is the limiting case for the Poisson distribution as μ becomes large.

Statistical treatment of random uncertainties distributed according to the Chi-square distribution

Chi-square test

N measurements of the same quantity $X \Rightarrow$ estimate $\bar{x}, \sigma_x$

Hypothesis : N measurements follow a Gaussian distribution

Divide the N measurements into bins

Bin	Observed	Expected $(N \times P_{Gauss})$
1	5	6
2	10	12
3	11	12
4	1	6

k ↓

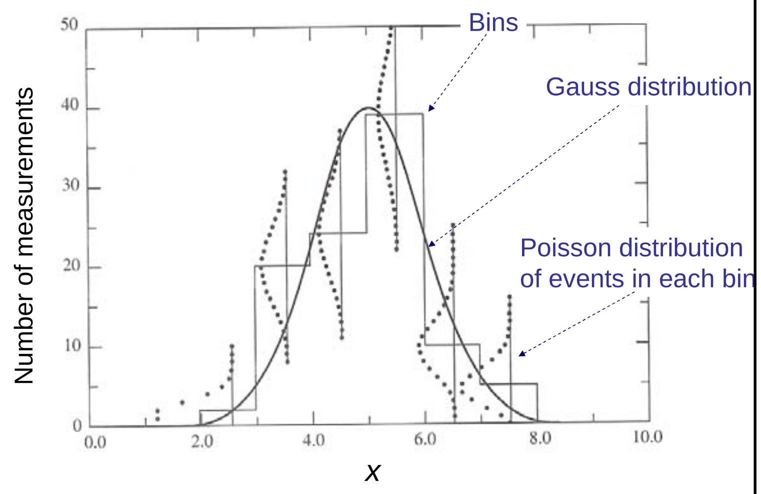
(see figure next slide)

Difference $Observed_k - Expected_k \rightarrow 0$ if Hypothesis is correct

$Observed_k$ is the result of a counting experiment : error $= \sqrt{Expected_k}$

Form the test : $\dfrac{(Observed_k - Expected_k)}{\sqrt{Expected_k}}$; $\chi^2 = \sum \dfrac{(Observed_k - Expected_k)^2}{Expected_k}$

χ^2 indicator of the agreement between the observed and expected distributions



Chi-square distribution

$$P_{Chi-square}(x,d) = \frac{(1/2)^{d/2}}{\Gamma(d/2)} x^{d/2-1} e^{-x/2}$$

where

d is the number of degrees of freedom

Γ is the Gamma function

Properties

$\bar{x} = d$

$\sigma = \sqrt{2d}$

Note: Degrees of freedom corresponds to the number of remaining choices

(1/2)

Chi-square: Summary

Test for the type of the parent distribution

$$\chi^2 = \sum_{k=1}^{n} \frac{(O_k - E_k)^2}{E_k}$$

where n is the number of bins

Agreement between observed and expected distributions :

$\chi^2 = 0$ Agreement perfect (unlikely to occur)

$\chi^2 \leq n$ Agreement acceptable

$\chi^2 >> n$ Significant disagreement

Chi-square: Summary

(2/2)

Reduced chi-square $\tilde{\chi}^2$

$$\tilde{\chi}^2 = \chi^2 / d$$

where d is the number of degrees of freedom $d = n - c$

n is the number of observations (number of bins)

c is the number of parameters to be calculated

Chi-square probability

$$P_{Chi-square}\left(\tilde{\chi}^2 \geq \tilde{\chi}^2_{observed}\right) < \text{X\% significance level}$$

reject the expected distribution at the X% significance level

(usually X% = 5% or 1%)

Uncertainty due to systematic errors

Uncertainty due to systematic error

Since systematic error is constant for repeated measurements, the uncertainty due to systematic error must be estimated.

The systematic uncertainty estimate can be based on:

- -Inter-laboratory test
- -Comparison against standards
- -Comparison of independent measurements (different principles)
- -Calibration reports
- -Engineering judgment

…

B_{95} is an estimate of the systematic error at the 95% confidence level

B_{95}

Note: 'B' stand for bias

Total uncertainty

The American Society of Mechanical Engineers (ASME) standard on measurement uncertainty

Total uncertainty

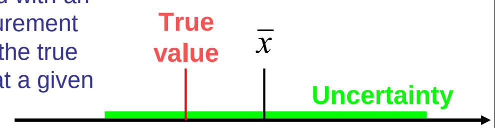
The total uncertainty in a measurement is the combination of

- uncertainty due to systematic error and
- uncertainty due to random error.

Uncertainty, U, is defined as an interval about the measured value, that has a pre-assigned probability of containing the true value:

$$\bar{x} \pm U_{95} \quad \text{with 95\% confidence}$$

Uncertainty is associated with an interval about the measurement mean $\bar{x}$, within which the true value is expected to lie at a given level of confidence



ASME Total uncertainty equation

Under the assumption that the systematic uncertainty component is normally distributed, the uncertainty U with 95% confidence is calculated by

$$U_{95} = \sqrt{B_{95}{}^2 + (t_{95}\sigma_{\bar{x}})^2}$$

where

B_{95} is an estimate of the systematic error at the 95% confidence level

$\sigma_{\bar{x}}$ is the standard deviation of the mean

$t_{95} = 2$ for normally distributed errors and large degrees of freedom ($N > 30$)

t_{95} definition:
Probability of 95% that a measurement falls within $t_{95}\sigma_{\bar{x}}$ of the true value

Uncertainty of a result

- Direct vs. indirect measurement
- Propagation of measurement uncertainties into a result
- Uncertainty for the uncertainty of a result

Direct vs. indirect measurement

Direct measurement:
The value of the unknown quantity is the measured quantity (e.g. length of an object)

Indirect measurement:
The value of the unknown quantity is obtained by calculation from other measured quantities (e.g. determination of the density of a body from its mass and volume $\rho = m / V$)

The result of an indirect measurement is expressed

$$R = f(\bar{x}_1, \bar{x}_2, ..., \bar{x}_J)$$

where $\bar{x}_i = 1/N_i \sum_{k=1}^{N_i} x_{i_k}$ is the mean of X_i

N_i the number of measurements of X_i

Propagation of uncertainties into a result: Independent quantities

The independent and normally distributed quantities $x_1,...,x_J$ are measured with standard deviations $\sigma_{\bar{x}_1},...,\sigma_{\bar{x}_J}$

A result R is calculated from the relation $R = f(\bar{x}_1,...,\bar{x}_J)$

The standard deviation of the result R is :

$$\sigma_R = \sqrt{\left(\frac{\partial f}{\partial x_1}\sigma_{\bar{x}_1}\right)^2 + ... + \left(\frac{\partial f}{\partial x_J}\sigma_{\bar{x}_J}\right)^2}$$

(Error Propagation Equation)

Propagation of uncertainty: Dependent quantities

A result R is defined as a function of the quantities $x_1,...,x_J$

$R = f(x_1,...,x_J)$

where $x_1,...,x_J$ are measured directly and have standard deviation $\sigma_{x_1},...,\sigma_{x_J}$

Two cases

1) R is expressed as a linear combination of $x_1,...,x_J$: $R = a_1x_1 + ... + a_Jx_J$

$$\sigma_R{}^2 = \sum_{i=1}^{J}\left((a_i\sigma_{x_i})^2 + 2\sum_{j=i+1}^{J} a_i a_j \sigma_{x_i x_j}\right)$$

2) R is expressed as a general function f : $R = f(x_1,...,x_J)$

$$\sigma_R{}^2 \cong \sum_{i=1}^{J}\left(\left(\frac{\partial f}{\partial x_i}\sigma_{x_i}\right)^2 + 2\sum_{j=i+1}^{J}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\sigma_{x_i x_j}\right)$$ (Approximation)

Propagation of uncertainty: Linear combination

$R = a_1x_1 + ... + a_Jx_J$

$$\sigma_R{}^2 = E\left((R - \bar{R})^2\right) = E\left(((a_1x_1 + ... + a_Jx_J) - (a_1\mu_1 + ... + a_J\mu_J))^2\right)$$

where $\mu_i = E(x_i) = 1/N\sum_{k=1}^{N} x_{i,k}$

Particular case of $J = 2$, $R = a_1x_1 + a_2x_2$

$$\sigma_R{}^2 = E\left(((a_1x_1 + a_2x_2) - (a_1\mu_1 + a_2\mu_2))^2\right) = E\left((a_1(x_1 - \mu_1) + a_2(x_2 - \mu_2))^2\right)$$

$$\sigma_R{}^2 = E\left((a_1(x_1 - \mu_1))^2\right) + E\left((a_2(x_2 - \mu_2))^2\right) + 2a_1a_2\underbrace{E((x_1 - \mu_1)(x_2 - \mu_2))}_{\neq 0 \text{ dependent quantities}}$$

$$\sigma_R{}^2 = (a_1\sigma_{x_1})^2 + (a_2\sigma_{x_2})^2 + 2a_1a_2\sigma_{x_1x_2}$$

General equation $\sigma_R{}^2 = \sum_{i=1}^{J}\left((a_i\sigma_{x_i})^2 + 2\sum_{j=i+1}^{J} a_i a_j \sigma_{x_i x_j}\right)$

For independent quantities $\sigma_{x_i x_j} = 0$ and $\sigma_R{}^2 = \sum_{i=1}^{J}\left((a_i\sigma_{x_i})^2\right)$

Propagation of uncertainty: General function

$R = f(x_1,...,x_J)$

$\sigma_R{}^2 = E\left((f(x_1,...,x_J) - f(\mu_1,...,\mu_J))^2\right)$

Taylor development of R in the neighborhood of $\mu_1,...,\mu_J$

$$R = f(\mu_1,...,\mu_J) + \frac{\partial f}{\partial x_1}(x_1 - \mu_1) + ... + \frac{\partial f}{\partial x_J}(x_J - \mu_J) + \underbrace{\text{higher order terms}}_{\text{neglected}}$$

$$\sigma_R{}^2 \cong E\left(\left(\frac{\partial f}{\partial x_1}(x_1 - \mu_1) + ... + \frac{\partial f}{\partial x_J}(x_J - \mu_J)\right)^2\right)$$ *Approximation!*

$$\sigma_R{}^2 \cong \left(\frac{\partial f}{\partial x_1}\sigma_{x_1}\right)^2 + ... + \left(\frac{\partial f}{\partial x_J}\sigma_{x_J}\right)^2 + 2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\sigma_{x_1x_2} + 2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_3}\sigma_{x_1x_3} + ... +$$

$$\sigma_R{}^2 \cong \sum_{i=1}^{J}\left(\left(\frac{\partial f}{\partial x_i}\sigma_{x_i}\right)^2 + 2\sum_{j=i+1}^{J}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\sigma_{x_i x_j}\right)$$

For independent quantities $\sigma_{x_i x_j} = 0$ and $\sigma_R{}^2 \cong \sum_{i=1}^{J}\left(\frac{\partial f}{\partial x_i}\sigma_{x_i}\right)^2$

Uncertainty of a result (independent quantities)

Single test (One test is conducted with the same instruments)

Uncertainty due to random error

$$\sigma_R = \sqrt{\left(\frac{\partial f}{\partial x_1}\sigma_{\bar{x}_1}\right)^2 + \ldots + \left(\frac{\partial f}{\partial x_J}\sigma_{\bar{x}_J}\right)^2}$$

Uncertainty due to systematic error

$$B_R = \sqrt{\left(\frac{\partial f}{\partial x_1}B_1\right)^2 + \ldots + \left(\frac{\partial f}{\partial x_J}B_J\right)^2}$$

Total Uncertainty $U = t\sqrt{\left(\frac{B_R}{2}\right)^2 + \sigma_R{}^2}$

Multiple test (More than one test is conducted with the same instruments: *M* repeated tests)

Uncertainty due to random error

$$\sigma_{\bar{R}} = \sigma_R / \sqrt{M}$$

Uncertainty due to systematic error

$$B_R$$

Total Uncertainty $U = t\sqrt{\left(\frac{B_R}{2}\right)^2 + \sigma_{\bar{R}}{}^2}$

General treatment of non random uncertainties

Propagation of uncertainties

The independent quantities $x_1,\ldots,x_J$ are measured with uncertainties $\Delta x_1,\ldots,\Delta x_J$

Upper bound of the standard deviation of $f(x_1,\ldots,x_J)$ is $\sigma_f \le \left|\frac{\partial f}{\partial x_1}\right|\Delta x_1 + \ldots + \left|\frac{\partial f}{\partial x_J}\right|\Delta x_J$

Function $f(x_1,\ldots,x_J)$	Uncertainty	Fractional Uncertainty
$\sum_{i=1}^{J} x_i$	$\sum_{i=1}^{J} \Delta x_i$	$\frac{\sum_{i=1}^{J} \Delta x_i}{\sum_{i=1}^{J} x_i}$
$\prod_{i=1}^{J} x_i$	$\sum_{i=1}^{J} \frac{\Delta x_i}{x_i} \times \prod_{i=1}^{J} x_i$	$\sum_{i=1}^{J} \frac{\Delta x_i}{x_i}$
ax	$a\,\Delta x$	$\frac{\Delta x}{x}$
x^n	$nx^{n-1}\Delta x$	$n\frac{\Delta x}{x}$

These expressions are upper bounds

Covariance and correlation

Measure of the extent to which a set of points $\{(x_1,y_1),\ldots,(x_N,y_N)\}$ supports a linear relation between *x* and *y*

Covariance in error propagation

N pairs of data $(x_1, y_1),\ldots,(x_N, y_N)$ are measured to estimate $q(x, y)$

All uncertainties are small

$$q_i = q(x_i, y_i) \cong q(\bar{x}, \bar{y}) + \left(\frac{\partial q}{\partial x}\right)_{\bar{x},\bar{y}} (x_i - \bar{x}) + \left(\frac{\partial q}{\partial y}\right)_{\bar{x},\bar{y}} (y_i - \bar{y})$$

$$\Rightarrow \bar{q} = \frac{1}{N}\sum_{i=1}^{N} q_i = \frac{1}{N}\sum_{i=1}^{N}\left(q(\bar{x}, \bar{y}) + \left(\frac{\partial q}{\partial x}\right)_{\bar{x},\bar{y}} (x_i - \bar{x}) + \left(\frac{\partial q}{\partial y}\right)_{\bar{x},\bar{y}} (y_i - \bar{y})\right)$$

$$\bar{q} = q(\bar{x}, \bar{y}) \qquad \longrightarrow 0 \qquad \longrightarrow 0$$

$$\Rightarrow \sigma_q^2 = \frac{1}{N-1}\sum_{i=1}^{N}(q_i - \bar{q})^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(\left(\frac{\partial q}{\partial x}\right)_{\bar{x},\bar{y}} (x_i - \bar{x}) + \left(\frac{\partial q}{\partial y}\right)_{\bar{x},\bar{y}} (y_i - \bar{y})\right)^2$$

$$\sigma_q^2 = \left(\frac{\partial q}{\partial x}\right)^2 \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2 + \left(\frac{\partial q}{\partial y}\right)^2 \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{y})^2$$

$$+ 2\left(\frac{\partial q}{\partial x}\right)\left(\frac{\partial q}{\partial y}\right)\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})$$

$$\sigma_{xy} = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})$$ The covariance of x and y

$$\sigma_q^2 = \left(\frac{\partial q}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial q}{\partial y}\right)^2 \sigma_y^2 + 2\left(\frac{\partial q}{\partial x}\right)\left(\frac{\partial q}{\partial y}\right)\sigma_{xy}$$

This equation gives the standard deviation σ_q whether or not the measurements are independent or normally distributed.

If the measurements of x and y are independent,

$$\sigma_{xy} = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y}) \xrightarrow[N\to\infty]{} 0$$

so that $\sigma_q^2 = \left(\frac{\partial q}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial q}{\partial y}\right)^2 \sigma_y^2$

If the measurements of x and y are NOT independent,

$\sigma_{xy} \neq 0 \quad \Leftrightarrow$ The errors in x and y are said to be correlated

- An overestimate of x is always accompanied by an overestimate of y, and vice versa => $(x_i - \bar{x})(y_i - \bar{y}) > 0 \Rightarrow \sigma_{xy} > 0$
- An overestimate of x is always accompanied by an underestimate of y, and vice versa => $(x_i - \bar{x})(y_i - \bar{y}) < 0 \Rightarrow \sigma_{xy} < 0$

Upper bound of the error

$$\sigma_q^2 = \left(\frac{\partial q}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial q}{\partial y}\right)^2 \sigma_y^2 + 2\left(\frac{\partial q}{\partial x}\right)\left(\frac{\partial q}{\partial y}\right)\sigma_{xy}$$

Applying the Cauchy-Schwarz inequality $|\sigma_{xy}| \le \sigma_x \sigma_y$

$$\Rightarrow \sigma_q^2 \le \left(\frac{\partial q}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial q}{\partial y}\right)^2 \sigma_y^2 + 2\left(\frac{\partial q}{\partial x}\right)\left(\frac{\partial q}{\partial y}\right)\sigma_x \sigma_y$$

$$\sigma_q^2 \le \left(\left|\frac{\partial q}{\partial x}\right|\sigma_x + \left|\frac{\partial q}{\partial y}\right|\sigma_y\right)^2$$

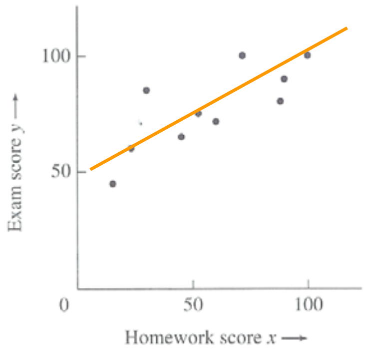
$$\sigma_q \le \left|\frac{\partial q}{\partial x}\right|\sigma_x + \left|\frac{\partial q}{\partial y}\right|\sigma_y$$

Proof for the upper bound $\delta q \le \left|\frac{\partial q}{\partial x}\right|\delta x + \left|\frac{\partial q}{\partial y}\right|\delta y$

Coefficient of linear correlation

The extent to which a set of measurements $(x_1,y_1),\ldots,(x_N,y_N)$ of two variables supports a linear relation between x and y is measured by the linear correlation coefficient:

$$r = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$



r	-1	0	+1
The points lie to a straight line	YES	NO	YES

Suppose that the two variables x and y satisfy a linear relation of the form:

$y = Ax + B$

$$\Rightarrow \begin{cases} y_i = Ax_i + B \\ \bar{y} = A\bar{x} + B \end{cases} \Rightarrow y_i - \bar{y} = A(x_i - \bar{x})$$

$$r = \frac{\sigma_{xy}}{\sigma_x \sigma_y} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}} = \frac{A}{|A|} = \pm 1$$

In General, $-1 \le r \le 1$

The Schwarz inequality $|\sigma_{xy}| \le \sigma_x \sigma_y$ implies

$$|r| = \left| \frac{\sigma_{xy}}{\sigma_x \sigma_y} \right| \le 1$$

Quantitative significance of r

The probability $\mathrm{Prob}_N(|r| \ge r_0)$ that N measurements of two uncorrelated variables x and y would produce a correlation coefficient with $|r| \ge r_0$

						r_o					
N	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
3	100	94	87	81	74	67	59	51	41	29	0
6	100	85	70	56	43	31	21	12	6	1	0
10	100	78	58	40	25	14	7	2	0.5		0
20	100	67	40	20	8	2	0.5	0.1			0
50	100	49	16	3	0.4						0

Example: The probability that 20 measurements (N=20) of two uncorrelated variables would yield $|r|$>0.5 is 2%. Thus, if 20 measurements gave r=0.5, we would have significant evidence of a linear correlation between the two variables (the correlation is significant at the 2% level)